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Linear representations of p-adic Heisenberg groups in spaces of analytic and continuous functions.

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ABSTRACT. Let K be a complete valued field extension of the field of p-adic numbers $\mathbb{Q}_p$. Let $\mathcal{D}$ be a closed unitary subring of the valuation ring Λ_K of K. Let $\mathcal{H}(3,\mathcal{D})$ be the 3-dimensional Heisenberg group with entries in $\mathcal{D}$. We shall give continuous linear representations of $\mathcal{H}(3,\mathcal{D})$ in the space $K < z >$ of restricted power series with cœfficients in K (= the Tate algebra in one variable, i.e. the space of analytic functions on Λ_K), analogous to Schrödinger representations of the classical Heisenberg group. On the other hand, assuming that $\mathcal{D}$ is compact, we shall obtain by the same way continuous linear representations of the profinite group $\mathcal{H}(3,\mathcal{D})$ in the space of continuous functions $\mathcal{C}(\mathcal{D},K)$, other analogues of Schrödinger representations. These representations are topologically irreducible. From the first representations, one obtains position and momentum bounded operators satisfying Heisenberg commutation relation and the Weyl algebra $A_1(K)$ as subalgebra of the algebra of bounded linear operators of $K < z >$. The closure $\widetilde{A}_1(K)$ of $A_1(K)$ is described.

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1. Introduction

Let A be a commutative unitary ring. By definition the Heisenberg group on A of order (dimension) 3 is the 3×3 unipotent upper triangular matrix group with entries in A, that is :

$$\mathcal{H}(3,A)=\Big\{s(a,b,c)=\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\ :\ a,b,c\in A\Big\}.$$

One has $s(0,0,0)=I_3$. For $s(a,b,c)$ and $s(a',b',c')$ in $\mathcal{H}(3,A)$, one sees that $s(a,b,c)s(a',b',c')=s(a+a',b+b',c+c'+ab')$ and $s(a,b,c)^{-1}=s(-a,-b,-c+ab)$.
The subset $\{s(a,0,c)/a,c\in A\}$ $\big[$resp.$\{s(0,b,c)/b,c\in A\}\big]$ is readily seen to be a subgroup of $\mathcal{H}(3,A)$ isomorphic to the additive group $A\times A$ $\big[$resp. isomorphic to the additive $A\times A$ and is a normal subgroup$\big]$ Also $\{s(a,0,0)/a,\in A\}$ is a subgroup isomorphic to the additive group A and $\mathcal{H}(3,A)=\{s(a,0,0)/a,c\in A\}\ltimes\{s(0,b,c)/b,c\in A\}$, with respect to the action $s(a,0,0)\cdot s(0,b,c)=s(0,b,c+ab)$.

Proposition 1.1.
(i) One has $s(a,b,c)=s(0,0,c)s(0,b,0)s(a,0,0)$.
(ii) The center of the group $\mathcal{H}(3,A)$ is equal to $\{s(0,0,c)/c\in A\}$
(iii) $s(a,0,0)s(0,0,c)=s(a,0,c)=s(0,0,c)s(a,0,0)$ and
$s(0,b,0)s(0,0,c)=s(0,b,c)=s(0,0,c)s(0,b,0)$
(iv) $s(a,b,c)s(a',b',c')s(a,b,c)^{-1}s(a',b',c')^{-1}=s(0,0,a'b-ab')$

Proof We only prove (ii). Assume that $s(a,b,c)$ belongs to $Z(\mathcal{H}(3,A))$ the center of $\mathcal{H}(3,A)$. Then for any $a',b',c'\in A$ one has $s(a+a',b+b',c+c'+ab')=s(a+a',b+b',c+c'+a'b)\Longrightarrow ab'=a'b,\forall a',b'\in A$. If $a'=0$ and $b'=1$, one has $a=0$ and $a'b=0,\forall a'\in A$ which implies $1.b=0$. It follows that the center $Z(\mathcal{H}(3,A))$ is contained in $\{s(0,0,c)/c\in A\}$. But this latest set is seen to be included in the center. Whence the equality. $\square$.

Let us notice that as a center, $Z(\mathcal{H}(3,A))$ is a normal subgroup. One immediately sees that the quotient group $\mathcal{H}(3,A)/Z(\mathcal{H}(3,A))$ is isomorphic to the product $A\times A$ of additve group.
One also deduces from (iv) that $Z(\mathcal{H}(3,A))=\{s(0,0,c)/c\in A\}$ is the group of commutators of $\mathcal{H}(3,A)$.

For finite commutative unitary ring as the quotient $\mathbb{Z}/m\mathbb{Z}$ and for finite fields the complex linear representations of these groups have been studied by many authors (see for instance [**6**], [**7**], [**12**])

In this talk, we consider a complete ultrametric valued field K, extension of the field of p-adic numbers $\mathbb{Q}_p$ and if $\mathcal{D}$ is a closed unitary subring of the valuation ring Λ_K of K, we are interested to the continuous linear representations of $\mathcal{H}(3,\mathcal{D})$ in appropriate function spaces.
The Schrödinger linear representations of the real Heisenberg group $\mathcal{H}(3,\mathbb{R})$ are obtained from their restriction on the center of $\mathcal{H}(3,\mathbb{R})$ equal the additive group of $\mathbb{R}$, restrictions which are the non

trivial characters of $\mathbb{R}$ (see for instance [**4**]). Our aim is to develop such theory for the group $\mathcal{H}(3,\mathcal{D})$ and then to find appropriate characters on the additive group of $\mathcal{D}$.
Let us notice that $\mathcal{H}(3,\mathcal{D})$ is a closed subgroup of the topological group of the general linear group $GL(3,\mathcal{D})$ on which one considers the topology induced by the norm $\|s\| = \max_{1\le i,j\le 3}|a_{i,j}|$ on the algebra of matrices $\mathrm{Mat}_3(\mathcal{D})$.

Since $\mathbb{Z}_p$ is contained in any closed unitary subring $\mathcal{D}$ of Λ_K, one obtains a way to find characters of $\mathcal{D}$ that extend some of $\mathbb{Z}_p$. Let us remind the following lemma.

Lemma 1.2. Assume that K is a complete valued field extension of $\mathbb{Q}_p$. The group $\widehat{\mathbb{Z}_p}$ of the continuous characters of the additive group Z_p in $K^\star$ corresponds bijectively to the principal unit group $D^-(1,1)$ of Λ_K.

Proof
For that, let us notice that if κ is a continuous character of $\mathbb{Z}_p$ in $K^\star$, then $|\kappa(1)| = 1$ and $\lim\limits_{n\to+\infty}\kappa(1)^{p^n} = \lim\limits_{n\to+\infty}\kappa(p^n) = \kappa(0) = 1$. Which implies $|\kappa(1)-1| < 1$,

Then there exists a positive integer m such that $|\kappa(1)^{p^m} - 1| < 1$. Which implies $|\kappa(1)-1| < 1$, $|\kappa(1)-1| < 1$, that is $\kappa(1)$ belongs to $D^-(1,1)$. and one obtains $\kappa(a) = \sum\limits_{n\ge 0}\binom{a}{n}(\kappa(1)-1)^n$, $\forall a \in \mathbb{Z}_p$.

Conversely let $q \in D^-(1,1)$, one immediately sees that for any $a \in \mathbb{Z}_p$, the series $q^a = \sum\limits_{n\ge 0}\binom{a}{n}(q-1)^n$ converges uniformly with respect to a, hence defines a continuous function of $\mathbb{Z}_p$ in K and defines a continuous character.

Corollary 1.3.
Let K be a complete valued field extension of $\mathbb{Q}_p$.
Any continuous character κ of $\mathbb{Z}_p$ in K is a strictly differentiable function with derivative $\kappa'(a) = \log(\kappa(1))\kappa(a)$.

Proof
This follows from the fact that $\lim\limits_{n\to+\infty} n|\kappa(1)-1|^n = 0$. Condition which as well known (cf [**9**] or [**11**]) implies that the function κ with Mahler expansion $\kappa(a) = \sum\limits_{n\ge 0}\binom{a}{n}(\kappa(1)-1)^n$ is strictly differentiable. A simple computation on the Mahler expansion gives the derivative. □

Application **The case when $|\kappa(1)-1| < |p|^{\frac{1}{p-1}}$**

That is $\kappa(1) = 1+\vartheta \in 1+E_p$, where $E_p = \{\beta \in K \ : \ |\beta| < |p|^{\frac{1}{p-1}}\}$ is the disc of convergence in K of the exponential. One has $\log(1+\vartheta) = \varpi \in E_p$, and $1+\vartheta = \exp(\varpi)$. Then for any positive integer m, $\kappa(m) = (1+\vartheta)^m = \exp(m\varpi)$. One obtains $\kappa(a) = \exp(\varpi a), \forall a \in \mathbb{Z}_p$. If $\varpi = 0$, one has the trivial character $\kappa_0(a) = 1$

Since $|\varpi| = |\log(1+\theta)| = |\theta| < |p|^{\frac{1}{p-1}}$, one has $\lim\limits_{n\to+\infty}\dfrac{|\varpi^n|}{|n!|} = 0$.

One concludes that the character κ such that $\kappa(a) = \exp(\varpi a) = \exp_{\varpi}(a) = \sum_{n\geq 0} \frac{\varpi^n}{n!} a^n$ is an *analytic function*, where $\varpi = \log(\kappa(1))$. Moreover one has $\kappa'(a) = \varpi\kappa(a)$.
In fact $\exp_{\varpi}$ *is a* restricted *power series with cœfficients in* K.

For any closed subring $\mathcal{D}$ of Λ_K, if t is an element of $\mathcal{D}$, then $|\varpi t| \leq |\varpi| < |p|^{\frac{1}{p-1}}$. It follows that κ extends to $\mathcal{D}$ by setting for $t \in \mathcal{D}$: $\kappa_{\varpi}(t) = \exp(\varpi t) = \exp_{\varpi}(t)$ which defines a character of the additive group $\mathcal{D}$ into $K^\star$. In particular $\exp_{\varpi}$ defines an analytic character of the additive group of Λ_K.

2. The case of a compact subring $\mathcal{D}$ of Λ_K

Let K be a complete field extension of $\mathbb{Q}_p$ and Λ_K its ring of valuation. We consider here a **compact** unitary subring $\mathcal{D}$ of Λ_K, for instance the ring $\mathbb{Z}_p$ of p-adic integers or the valuation ring of any finite extension of $\mathbb{Q}_p$ contained in K and vice versa.
The compact ring $\mathcal{D}$ being totally discontinuous one sees that the *Heisenberg group* $\mathcal{H}(3,\mathcal{D})$ *is a topological, totally discontinuous, compact group, that is a profinite group. For instance* $\mathcal{H}(3,\mathbb{Z}_p)$ *is a pro-p-group.*

Let us fix $0 \neq \varpi \in E_p$, the open disc of convergence of the p-adic exponential function. We have seen that the map $\exp_{\varpi} : t \longrightarrow exp_{\varpi}(t) = \sum_{n\geq 0} \frac{\varpi^n}{n!} t^n$ of Λ_K in $K^\star$ is a continuous character (even analytic) of the additive group Λ_K and by restriction a character of the additive group of any of its closed unitary subring.
Consider for the compact ring $\mathcal{D}$ the K-vector space $\mathcal{C}(\mathcal{D},K)$ of continuous functions f of $\mathcal{D}$ in K. With the supremum norm $\|f\|$ and the usual product of functions $\mathcal{C}(\mathcal{D},K)$ is a unitary Banach algebra.

Let $s = s(a,b,c)$ be an element of $\mathcal{H}(3,\mathcal{D})$ and f an element of $\mathcal{C}(\mathcal{D},K)$.
Set $\pi_s^{\varpi} f(t) = \exp_{\varpi}(bt+c) f(t+a)$.
It is clear that $\pi_s^{\varpi} f$ is a continuos function of $\mathcal{D}$ in K.

N.B.
If $\varpi = 0$, one has $\pi^0_{s(a,b,c)} f(t) = f(t+a)$. Then $\pi^0_{s(0,b,c)} f(t) = f(t)$ $\forall b,c \in \mathcal{D}$.

Lemma 2.1.
The map $(s,f) \longrightarrow \pi_s^{\varpi} f$ such that $\pi_s^{\varpi} f(t) = \exp_{\varpi}(bt+c) f(t+a)$ defines a continuous linear representation of $\mathcal{H}(3,\mathcal{D})$ in the Banach space $\mathcal{C}(\mathcal{D},K)$.

Proof
(1) Indeed, it is readily seen that for any $s \in \mathcal{H}(3,\mathcal{D})$, one has that π_s^{ϖ} is a continuous linear endomorphism of $\mathcal{C}(\mathcal{D},K)$ and each π_s^{ϖ} is an isometry.

For $s(a,b,c), s(a',b',c') \in \mathcal{H}(3,\mathcal{D}$ and f a continuous functions of $\mathcal{D}$ in K, It is a routine to verify that $\pi^{\varpi}_{s(a,b,c)s(a',b',c')}f(t) = \pi^{\varpi}_{s(a+a',b+b',c+c'+ab')}f(t) = pi^{\varpi}_{s(a,b,c)} \circ \pi^{\varpi}_{s(a',b',c')}f(t)$.

(2) Furthermore, $|\pi^{\varpi}_{s(a,b,c)}f(t) - \pi^{\varpi}_{s(a',b',c')}f(t)| \leq$
$\leq \max(|\exp_{\varpi}(bt+c) - \exp_{\varpi}(b't+c')||f(t+a)|, |\exp_{\varpi}(b't+c')||f(t+a) - f(t+a')|) \leq$
$\max(|\varpi((b-b')t + (c-c'))|\|f\|, |f(t+a) - f(t+a')|) \leq$
$\max(\max(|b-b'|, |c-c'|)\|f\|, |f(t+a) - f(t+a')|)$.
Since $\mathcal{D}$ is a metric compact space, continuity of functions implies uniform continuity. Hence for $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that if $|\theta - \theta'| < \eta_\varepsilon$ then $|f(\theta) - f(\theta')| < \varepsilon$. It follows that for $|a-a'| = |a+t-(t+a')| < \eta_\varepsilon$, one has $|f(t+a) - f(t+a')| < \varepsilon, \forall t \in \mathcal{D}$. According to the above inequality, if $\|s(a,b,c) - s(a',b',c')\| \leq \min(\varepsilon/\|f\|, \eta_\varepsilon)$, one sees that
$|\pi_{s(a,b,c)}f(t) - \pi_{s(a',b',c')}f(t)| < \varepsilon, \forall t \in \mathcal{D}$, which implies that $\|\pi_{s(a,b,c)}f - \pi_{s(a',b',c')}f\| < \varepsilon$. It follows that the representation π is continuous. □

The tangent operators

(a) $\varpi \neq 0$

By definition, one has $\pi_{s(a,0,0)}f(t) = f(t+a) = \tau_a f(t)$
$\pi_{s(0,b,0)}f(t) = \exp_{\varpi}(bt)f(t)$ and $\pi_{s(0,0,c)}f(t) = \exp_{\varpi}(c)f(t)$.

(b)

Let us consider the quotients :

(i) $\quad \Delta_a f(t) = \dfrac{f(t+a) - f(t)}{a}, \ a \neq 0$

(ii) $\quad M_b f(t) = \dfrac{exp_{\varpi}(bt)f(t) - f(t)}{b} = \dfrac{\exp_{\varpi}(bt) - 1}{b} f(t), \ b \neq 0$ and

(iii) $\quad \eta_c f(t) = \dfrac{\exp_{\varpi}(c)f(t) - f(t)}{c} = \dfrac{exp_{\varpi}(-c) - 1}{c} f(t), \ c \neq 0$.

$(i)'$ If a tends towards 0, since there exist continuous functions on $\mathcal{D}$ that are not derivable, the limit $\lim\limits_{a \to 0} \dfrac{f(t+a) - f(t)}{a}$ does not always exists.
However this limit exists for (strictly) differentiable functions and defines an unbounded linear operator of $\mathcal{C}(\mathcal{D}, K)$ whose domain contains the space of strictly differentiable functions of $\mathcal{D}$ in K.
But, on the other hand the following limits exist and are uniform limits

$(ii)'$ $\quad \lim\limits_{b \to 0} M_b f(t) = \lim\limits_{b \to 0} \dfrac{\exp_{\varpi}(bt) - 1}{b} f(t) = \varpi t f(t)$

$(iii)'$ $\quad \lim\limits_{c \to 0} \dfrac{\exp_{\varpi}(c)f(t) - f(t)}{c} = \lim\limits_{c \to 0} \dfrac{\exp_{\varpi}(c) - 1}{c} f(t) = \varpi f(t)$.

We are ready for the statement of the following ultrametric counterpart of classical Schrödinger representations.

Theorem 2.2. *Assume that $\mathcal{D}$ is a compact unitary subring of the valuation subring of K.*
Consider ϖ a non zero element of E_p.
Then the continuous linear representation $(\mathcal{H}(3,\mathcal{D}),\ \pi^{\varpi},\ \mathcal{C}(\mathcal{D}, K))$ is topologically irreducible

Proof

Let W be a closed invariant linear subspace of $\mathcal{C}(\mathcal{D}, K)$.

$-(a)-$ One sees that W is stable by the quotient maps M_b and by passing to limit it is stable by the $\varpi m(f)$ where $m(f)(t) = tf(t)$. By linearity and density of the set of polynomial functions [by Stone-Weierstrass-Kaplansky theorem, see for instance [**10**]], one obtains that for any continuous function g of $\mathcal{D}$ in K and any $f \in W$, gf belongs to W. In other words W is an ideal of $\mathcal{C}(\mathcal{D}, K)$.

$-(b)-$

Let $f \in W, f \neq 0$ and let us consider $0 < \varepsilon' < \|f\|$, then there exist $t_{\varepsilon'} \in \mathcal{D}$ such that $0 < \varepsilon = \|f\| - \varepsilon' < |f(t_{\varepsilon'})|$. Therefore $O_\varepsilon = \{t \in \mathcal{D}/|f(t)| > \varepsilon\}$ is an open and closed non empty subset of $\mathcal{D}$. Let h_ε be the function such that $h_\varepsilon(t) = \dfrac{1}{f(t)}$ if $t \in O_\varepsilon$ and $h_\varepsilon(t) = 0$ otherwise. It is a continuous function such that $h_\varepsilon f = \chi_{O_\varepsilon}$, the characteristic function of O_ε [cf. [**10**], Proof of Theorem 6.27]. Since W is an ideal, $\chi_{O_\varepsilon} = h_\varepsilon f$ belongs to W. On the other hand $\pi_{s(-a,0,0)}\chi_{O_\varepsilon} = \tau_{-a}\chi_{O_\varepsilon} = \chi_{a+O_\varepsilon} \in W,\ \forall a \in \mathcal{D}$. One sees that $\mathcal{D} = \bigcup_{a\in\mathcal{D}} (a + O_\varepsilon)$.

Since $\mathcal{D}$ is compact, one has a finite covering $\mathcal{D} = \bigcup_{1\leq j\leq\nu} (a_j + O_\varepsilon)$.

Applying the inclusion-exclusion formula for characteristic functions one sees that $1 = \chi_{\mathcal{D}} = \chi_{\cup_{1\leq 1\leq\nu}(a_i+O_\varepsilon)} = \sum_{j=1}^{\nu} \chi_{a_j+O_\varepsilon} + \sum_{k=2}^{\nu}(-1)^{k-1} \sum_{1\leq j_1<j_2<\cdots<j_k\leq\nu} \chi_{a_{j_1}+O_\varepsilon} \cdots \chi_{a_{j_k}+O_\varepsilon}$ belongs to the ideal W, as any $\chi_{a_j+O_\varepsilon}$ does. Therefore $W = \mathcal{C}(\mathcal{D}, K)$.

We have finished proving that the representation π^ϖ is topologically irreducible. $\square$

We have putted π^ϖ the linear representation associated to $\varpi \in E_p \setminus \{0\}$ such that $\pi^\varpi_{s(a,b,c)} f(t) = \exp_\varpi(bt + c) f(t + a)$.

Corollary 2.3.

(i) Let $\varpi_1,\ \varpi_2 \in E_p \setminus \{0\}$.
Then the representations π^{ϖ_1} and π^{ϖ_2} are equivalent if and only if $\varpi_1 = \varpi_2$

(ii) Let $c \in \mathcal{D}$, then $\exp(c\varpi) \cdot id$ is an intertwining operator of the representation π^ϖ. If φ is an intertwining operator of the representation π^ϖ, then $\varphi = \varphi(1)id$, with $\varphi(1)$ a constant in K. (Schur Lemma)

Proof

(i) is easy

(ii) Let φ be an intertwining operator of the representation π^ϖ, that is $\pi^\varpi_s \circ \varphi = \varphi \circ \pi^\varpi_s,\ \forall s \in \mathcal{H}(3, \mathcal{D})$. In particular $\pi^\varpi_{s(0,b,0)} \circ \varphi(f) = \exp_{\varpi b} \varphi(f) = \varphi(\exp_{\varpi_2 b} f)$. Hence for $b \neq 0$, one has $M_b\varphi(f) = \varphi(M_b f)$. When b tends towards 0, one has $M_b(t) \to \varpi t$. Hence $\varpi t\varphi(f) = \varphi(\varpi t f) \Longrightarrow t\varphi(f) = \varphi(tf)$.

From what one deduces that $\varphi(gf) = g\varphi(f) \Longrightarrow \varphi(g) = \varphi(1)g$. That is $\varphi = \varphi(1)id$.

Moreover since $\tau_a \circ \varphi = \varphi \circ \tau_a \Longrightarrow \tau_a\varphi(f) = \varphi(\tau_a f) = \varphi(1)\tau_a f$, one has $\tau_a\varphi(1) = \varphi(1)\tau_a 1. = \varphi(1)$. That is $\varphi(1)(x + a) = \varphi(1)(x), \forall a, x \in \mathcal{D}$. Hence $\varphi(1)(a) = \varphi(1)(0), \forall a \in \mathcal{D}$. That is $\varphi(1)$ is a constant function, element of the field K $\square$

Scholie

(α) The space $\mathcal{C}^1(\mathcal{D}, K)$ of strictly differentiable functions of $\mathcal{D}$ in K is a subspace of $\mathcal{C}(\mathcal{D}, K)$ invariant by any representation π^{ϖ}. Which with its own topology is topologically irreducible although it is a dense subspace of the space of continuous functions. Any strictly differentiable function f has a derivative f' that is a continuous function not necessary strictly differentiable. Then the operator of derivation is an unbounded operator that domain contains $\mathcal{C}^1(\mathcal{D}, K)$ with values in $\mathcal{C}(\mathcal{D}, K)$.

(β) The space $\mathcal{A}(\mathcal{D}, K)$ of analytic functions of $\mathcal{D}$ in K is another subspace of $\mathcal{C}(\mathcal{D}, K)$ that is a non closed subspace invariant by π^{ϖ}. We will be concerned with such representation in the sequel.

(γ) • The case when $\mathcal{D} = \mathbb{Z}_p$ can be of particular interest.
Indeed we have described all the continuous characters κ of $\mathbb{Z}_p$ in $K^{\star}$. To any character $\kappa \in \widehat{\mathbb{Z}_p}$ one can associate a continuous linear representation π^{κ} of $\mathcal{H}(3, \mathbb{Z}_p)$ in the space $\mathcal{C}(\mathbb{Z}_p, K)$ by setting for any continuous function and any $s = s(a, b, c) \in \mathcal{H}(3, \mathbb{Z}_p)$: $\pi^{\kappa}_s f(t) = \kappa(bt + c) f(t + a)$. Unless $\kappa(1)$ is a p^{ν}-root of unity in K, what is said for the above representation associated to an analytic character remains mutatis mutandis true.
• If $\kappa(1)$ is a p^{ν}-root of unity, then the subspace of locally constant functions $\mathcal{C}(\mathbb{Z}_p, K)^{p^{\nu}\mathbb{Z}_p} = \{f : \mathbb{Z}_p \longrightarrow K \ / \ f(t + t') = f(t), \forall t' \in p^{\nu}\mathbb{Z}_p\}$ is invariant by π^{κ}. On can show that the restriction of π^{κ} to this subspace is a finite dimensional irreducible linear representation.

(δ) **The one parameter subgroups associated to the representation** π^{ϖ}.

The representation π^{ϖ} is not smooth.
However one has the following one parameter subgroups associated to π^{ϖ}.
That is group homomorphisms of $\mathcal{D}$ in the group $Aut(\mathcal{C}(\mathcal{D}, K))$ of the linear automorphisms of the Banach space $\mathcal{C}(\mathcal{D}, K)$.

(δ_1) The first is defined by the map $a \longrightarrow \tau_a$, which is not a smooth one parameter group.

(δ_2) The second is the map $b \longrightarrow \pi^{\varpi}_{s(0,b,0)}$. This one parameter group is smooth and if one considers the linear operator m defined by setting $m(f)(t) = tf(t)$, one has for any element $b \in \mathcal{D}$ the linear automorphism $\exp_{\varpi b}(m) = \exp_{\varpi}(bm) = \sum_{n \geq 0} \frac{\varpi^n b^n}{n!} m^n$ of $\mathcal{C}(\mathcal{D}, K)$ and one has $\exp_{\varpi}(bm) f =$ $= \exp_{\varpi b} \cdot f$ for any continuous function f of $\mathcal{D}$ in K

(δ_3) The third is given by the smooth character $\exp_{\varpi} : c \longrightarrow \pi^{\varpi}_{s(0,0,c)}$. $\square$

Notice that one has $\tau_a \circ m - m \circ \tau_a = a\tau_a, \forall a \in \mathcal{D}$.

3. Analytic representations

In this section we consider a non necessary compact, closed unitary subring $\mathfrak{D}$ of the valuation ring $\Lambda_K = \Lambda$ of the complete valued field K extension of the field of p-adic numbers $\mathbb{Q}_p$. We have noticed that considering the space of analytic fucntions $\mathcal{A}(\mathfrak{D}, K)$, if $\exp_{\varpi}$ is an analytic character of $\mathbb{Z}_p$, then one can defines a linear representation $U = U^{\varpi}$ of the Heisenberg group $\mathcal{H}(3, \mathfrak{D})$ in $\mathcal{A}(\mathfrak{D}, K)$ such that if $s = s(a, b, c)$ is an element of $\mathcal{H}(3, \mathfrak{D})$ and f an analytic function $\mathfrak{D}$ in K, then one has $U_{s(a,b,c)}f(t) = \exp_{\varpi}(bt + c)f(t + a)$. The space $\mathcal{A}(\mathfrak{D}, K)$ is complete with respect to the Gauss norm, but if the field K is of discrete valuation the Gauss norm differs from the uniform norm, we consider $\mathcal{A}(\mathfrak{D}, K)$ rather as the Tate algebra in one indeterminate, that is the subalgebra $K < z >$ of the algebra of formal power series whose elements are the formal power series $f(z) = \sum_{n\geq 0} a_n z^n$ such that $\lim_{n\to+\infty} |a_n| = 0$. With the Gauss norm $\|f\| = \sup_{n\geq 0} |a_n|$, the algebra $K < z >$ becomes an ultrametric unitary algebra with a multiplicative norm. The elements of $K < z >$ are also called the restricted power series with cœfficients in K.

3.1. Substitution in restricted power series.

Let $K[[X]]$ be the ring of formal power series with cœfficients in K. For $f = \sum_{n\geq 0} a_n X^n \in K[[X]]$, one has in $K[[X, Y]] = K[[X]][[Y]]$ the formal Taylor expansion $f(X + Y) = \sum_{n\geq 0} a_n(X + Y)^n = \sum_{j\geq 0} f^{[j]}(X)Y^j$:

where $f^{[j]}(X) = \sum_{i\geq 0} \binom{i+j}{i} a_{i+j}X^i$. One has $f^{[1]}(X) = \sum_{i\geq 0}(i + 1)a_{i+1}X^i = f'(X)$ the formal derivative of f and if the field K is of characteristic 0, one sees that $f^{[j]}(X) = \dfrac{f^{(j)}(X)}{j!}$, where $f^{(j)}$ is the j^{th}-derivative of f.

Now let $f = \sum_{n\geq 0} a_n z^n \in K < z >$. For any integer j , one sees that $f^{[j]}(z) = \sum_{i\geq 0} \binom{i+j}{i} a_{i+j}z^i$ belongs to $K < z >$, with $\|f^{[j]}\| = \sup_{i\geq 0} \left|\binom{i+j}{i}\right| |a_{i+j}| \leq \sup_{i\geq 0} |a_{i+j}|$.
Since $\lim_{j\to+\infty} |a_j| = 0$, one has $\lim_{j\to+\infty} \sup_{i\geq 0} |a_{i+j}| = \limsup_{j\to+\infty} |a_j| = 0$ which implies $\lim_{j\to+\infty} \|f^{[j]}\| = 0$.

Let $h(z) = \sum_{n\geq 0} b_n z^n = b_0 + g(z) \in K < z >$ be such that $\|h\| = \sup_{n\geq 0} |b_n| = \max |b_0|, \|g\|) \leq 1$.

For the integers $i,j \geq 0$, one has $\left|\binom{i+j}{i}\right| |a_{i+j}| \|g^i\| \leq |a_{i+j}| \|g\|^i \leq |a_{i+j}|$.
Hence $\lim\limits_{i \to +\infty} \left|\binom{i+j}{i}\right| |a_{i+j}| \|g^i\| = 0, \forall j \geq 0$ fixed, and one has the convergent sum of restricted power series $\sum\limits_{i \geq 0} \binom{i+j}{i} a_{i+j} g(z)^i, \forall j \geq 0$, that is for any integer $j \geq 0$ the power series $\sum\limits_{i \geq 0} \binom{i+j}{i} a_{i+j} g(z)^i$ belongs to $K < z >$.

Since $g(0) = 0$, one has by substitution of formal power series that $f^{[j]} \circ g(z) = \sum\limits_{i \geq 0} \binom{i+j}{i} a_{i+j} g(z)^i$ belongs to $K < z >$.

Moreover $\|f^{[j]} \circ g\| \leq \sup\limits_{i \geq 0} \left|\binom{i+j}{i}\right| |a_{i+j}| \|g^i\| \leq \sup\limits_{i \geq 0} \left|\binom{i+j}{i}\right| |a_{i+j}| = \|f^{[j]}\|, \forall j \geq 0$.

On the other hand, since $|b_0| \leq 1$, one has $|b_0|^j \|f^{[j]} \circ g\| \leq \|f^{[j]}\|$. One then deduces that $\lim\limits_{j \geq 0} |b_0|^j \|f^{[j]} \circ g\| = 0$ and one obtains the convergent sum of restricted power series

$$\sum_{j \geq 0} b_0^j f^{[j]} \circ g(z) = f(b_0 + g(z)) = f(h(z)), \text{ an element of } K < z >.$$

In particular for α and β elements of the valuation ring of K; one has an element of $K < z >$ defined by setting $f(\alpha z + \beta) = \sum\limits_{j \geq 0} \beta^j f^{[j]}(\alpha z)$.

3.2. Linear representations of $\mathcal{H}(3, \mathfrak{D})$ in $K < z >$.

For $\alpha, \beta \in \mathfrak{D}$, we have seen that one can substitute $\alpha z + \beta$ in f obtaining again an element of $K < z >$ such that
$f(\alpha z + \beta) = \sum\limits_{j \geq 0} \beta^j f^{[j]}(\alpha z)$ and $f^{[j]}(\alpha z) = \sum\limits_{i \geq 0} \binom{i+j}{i} a_{i+j} \alpha^i z^i$.

Let us remind that if $|\varpi| < |p|^{\frac{1}{p-1}}$, then the series $\exp_\varpi(z) = \sum\limits_{n \geq 0} \frac{\varpi^n}{n!} z^n$, is a non constant restricted power series for $\varpi \neq 0$, that is a non constant element of the Tate algebra $K < z >$. Moreover $\|\exp_\varpi\| = \sup\limits_{n \geq 0} \frac{|\varpi|^n}{|n!|} = 1$.

Now, let $s = s(a,b,c)$ be an element of $\mathcal{H}(3, \mathfrak{D})$, the Heisenberg group with entries in the closed unitary subring $\mathfrak{D}$ of the valuation Λ of the complete valued field K extension of $\mathbb{Q}_p$.
For $f \in K < z >$, let us set $U^\varpi_{s(a,b,c)} f(z) = \exp_\varpi(bz + c) f(z + a)$.
One obtains by the way an element $U^\varpi_{s(a,b,c)} f$ of $K < z >$.
It is obvious that $U^\varpi_{s(a,b,c)}$ is a continuous linear endomorphism of $K < z >$.
One verifies as already done above that $U^\varpi_{s(a,b,c)s(a',b',c')} = U^\varpi_{s(a,b,c)} \circ U^\varpi_{s(a',b',c')}$.

Hence the map $U^{\varpi} : \mathcal{H}(3, \mathfrak{D}) \longrightarrow \mathcal{L}(K < z >)$ such that
$U^{\varpi}_{s(a,b,c)} f(z) = \exp_{\varpi}(bz + c) f(z + a)$ is a linear representation.

Since $\exp_{\varpi}(bz+c) = \sum_{n\geq 0} \frac{\varpi^n}{n!}(bz+c)^n$, with $\|(bz+c)^n\| = \|bz+c\|^n = \max(|b|,|c|)^n \leq 1$, one has $\frac{|\varpi|^n}{|n!|}\|(bz+c)^n\| \leq \frac{|\varpi|^n}{|n!|} < |p|^{n(\nu - \frac{1}{p-1})} < 1, \forall n \geq 1$ and then $\|\exp_{\varpi}(bz + c)\| = 1$.

On the other hand one has $U^{\varpi}_{s(a,0,0)} f(z) = f(z + a) = \tau_a f(z) = \sum_{j\geq 0} a^j f^{[j]}(z)$, then $\|U^{\varpi}_{s(a,0,0)} f\| \leq \sup_{j\geq 0} |a|^j \|f^{[j]}\| \leq \sup_{j\geq 0} \|f^{[j]}\| \leq \|f\|$. In the same way, one has $\|U^{\varpi}_{s(-a,0,0)} f\| \leq \|f\|$. One then sees that $\|U^{\varpi}_{s(a,0,0)} f\| = \|f\|, \forall a \in \mathcal{D}$.
It follows that $\|U^{\varpi}_{s(a,b,c)} f\| = \|\exp_{\varpi}(bz + c)\| \|U^{\varpi}_{s(a,0,0)} f\| = \|f\|$.
The representation is then said to be unitary.

Lemma 3.1. *The linear representation $(\mathcal{H}(3, \mathfrak{D}),\ U^{\varpi},\ K < z >)$ is continuous.*

Proof

For $a \in \mathfrak{D}$ and $f \in K < z >$, one has $\tau_a f = \sum_{j\geq 0} a^j f^{[j]}$. One verifies that $\|\tau_a f - \tau_{a'} f\| \leq \sup_{j\geq 1} |a^j - a'^j| \|f^{[j]}\| \leq |a - a'| \sup_{j\geq 1} \|f^{[j]}\| \leq |a - a'| \|f\|$.
On the other hand $\|\exp_{\varpi}(bz+c) - \exp_{\varpi}(b'z+c')\| \leq \sup_{n\geq 1} \left|\frac{\varpi^n}{n!}\right| \|(bz + c)^n - (b'z + c')^n\|$
$\leq \sup_{n\geq 1} \left|\frac{\varpi^n}{n!}\right| \cdot \max(|b - b'|, |c - c'|) \leq \max(|b - b'|, |c - c'|)$
Let $s(a, b, c)$ and $s(a', b', c')$ be two elements of the Heisenberg group, one obtains
$\|U^{\varpi}_{s(a,b,c)} f - U^{\varpi}_{s(a',b',c')} f\| \leq$
$\leq \max(\|\exp_{\varpi}(bz + c)\| \|\tau_a f - \tau_{a'} f\|, \|\exp_{\varpi}(bz + c) - \exp_{\varpi}(b'z + c')\| \|\tau_{a'} f\|) =$
$= \max(\|\tau_a f - \tau_{a'} f\|, \|\exp_{\varpi}(bz + c) - \exp_{\varpi}(b'z + c')\|)\|f\|$
Summarizing we have $\|U^{\varpi}_{s(a,b,c)} f - U^{\varpi}_{s(a',b',c')} f\| \leq \max(|a - a'|, |b - b'|, |c - c'|)\|f\| =$
$= \|s(a, b, c) - s(a', b', c')\| \|f\|$ and the map $s(a, b, c) \longrightarrow U^{\varpi}_{s(a,b,c)} f$ is continuous.
□

The linear representation U^ϖ is smooth

By definition, one has $U^\varpi_{s(a,0,0)}f(z) = f(z+a) = \tau_a f(z)$
$U^\varpi_{s(0,b,0)}f(z) = \exp_\varpi(bz)f(z)$ and $U^\varpi_{s(0,0,c)}f(z) = \exp_\varpi(c)f(z)$.
As previously, let us consider in $K < z >$, for $a \neq 0, b \neq 0$ and $c \neq 0$, the quotients :

$$(i) \quad \varDelta_a(f) = \frac{\tau_a f - f}{a}$$

$$(ii) \quad M_b(f) = \frac{exp_\varpi(bz)f - f}{b} = \frac{\exp_\varpi(bz) - 1}{b} f \text{ and}$$

$$(iii) \quad \eta_c(f) = \frac{\exp_\varpi(c)f - f}{c} = \frac{exp_\varpi(c) - 1}{c} f.$$

Since $\tau_a f = \sum_{j\geq 0} a^j f^{[j]} = f + af' + \sum_{j\geq 2} a^j f^{[j]}$, one immediately sees that
$\varDelta_a(f) = f' + a\sum_{j\geq 2} a^{j-2} f^{[j]}$ and $\|\varDelta_a(f) - f'\| \leq |a|\|f\| \Longrightarrow \sup_{f\neq 0} \frac{\|\varDelta_a(f) - f'\|}{\|f\|} \leq |a|$.
It follows that in $\mathcal{L}(K < z >)$ one has:

$$\lim_{a\to 0} \varDelta_a = \partial, \text{ where } \partial(f) = f'. \text{ Furthermore } \|\partial\| = 1. \qquad \mathbf{(i)'}$$

In the same way, one obtains $M_b(f) = \frac{exp_\varpi(bz)f - f}{b} = \varpi z{\cdot}f + b\left(\sum_{n\geq 2} \frac{\varpi^n}{n!} b^{n-2} z^n\right) f.$
with $\|M_b(f) - \varpi z \cdot f\| < |b|\|f\|$, and $\lim_{b\to 0} \sup_{\|f\|\neq 0} \frac{\|M_b(f) - \varpi z \cdot f\|}{\|f\|} = 0$. That is, in $\mathcal{L}(K < z >)$, one has:

$$\lim_{b\to 0} M_b = \varpi m_z \text{ where } m_z(f) = zf \text{ and } \|m_z(f)\| = \|f\| \qquad \mathbf{(ii)'}\ .$$

The formulas (i)' and (ii)' are used in the proof of the forthcoming theorem.
Obviously $\eta_c(f) = \frac{\exp_\varpi(c)f - f}{c} = \varpi f + c\left(\sum_{n\geq 2} \frac{\varpi^n}{n!} c^{n-2}\right) f.$
Then $\|\eta_c(f) - \varpi f\| \leq |c| \sup_{n\geq 2} \frac{|\varpi^n|}{|n!|}\|f\| < |c|\|f\|$ and one obtains $\lim_{c\to 0} \eta_c = \varpi{\cdot}id$. $\quad \mathbf{(iii)'}$

Theorem 3.2. *Assume that $0 \neq \varpi \in E_p$.*
Then the continuous linear representation $(\mathcal{H}(3, \mathfrak{D}),\ U^\varpi,\ K < z >)$ is topologically irreducible.

Proof

Let W be a closed linear subspace of $K < z >$ invariant by the representation U^{ϖ}.

(1) Let f be an element of W, for any $a \in \mathfrak{D}$, one has $\tau_a f = U_{s(a,0,0)} f \in W$. Hence $\Delta_a(f) = \dfrac{\tau_a f - f}{a}$ belongs to W.
Since W is a closed linear subspace, one sees that $\lim\limits_{a \to 0} \Delta_a f = f' = \partial(f)$ also belongs to W.
Hence W is stable by the derivative operator ∂ and for any integer $n \geq 0$, one has $\partial^{\circ n}(W) \subset W$.

(2) In the same way, W is stable by M_b, $\forall b \in \mathfrak{D}$ and then it is stable by the limit $\lim\limits_{b \longrightarrow 0} M_b(f) = \varpi m_z(f)$, where $m_z(f) = zf$. Hence $z^n W \subset W, \forall n \geq 0$. Therefore by linearity and continuity, for any $g \in K < z >$ and for any $f \in W$, one has $gf \in W$. That is W is an ideal of $K < z >$.

(3) Assume that $W \neq 0$. Let $f \in W, f \neq 0$. According to Weierstrass preparation theorem (see for instance [**5**]) there exists a polynomial P and a restricted power series g such that $f = Pg$, with $\|g - 1\| < 1$, therefore g is invertible in $K < z >$. It follows that $P = g^{-1} f$ belongs to W.
Let ν be the degree of the polynomial P, then $\partial^{\circ \nu} P = \nu! a_{\nu} \in W$ and the formal power series 1 belongs to W. It follows that the nonzero ideal W is equal to $K < z >$.
Therefore the linear representation U^{ϖ} is topologically irreducible. □

PROPOSITION 3.3.
The algebra $End_{U^{\varpi}}(K < z >)$ of the continuous linear intertwining operators of the representation U^{ϖ} is equal to $K.id$, where id is the identity map of $K < z >$.
(Schur lemma)

Proof

Let φ be a continuous linear endomorphism of $K < z >$ such that
$\varphi \circ U^{\varpi}_{s(a,b,c)} = U^{\varpi}_{s(a,b,c)} \circ \varphi, \forall s(a,b,c) \in \mathcal{H}(3, \mathfrak{D})$.
One immediately sees, on one hand that $\varphi \circ \Delta_a = \Delta_a \circ \varphi, \forall a \in \mathfrak{D} \Longrightarrow \varphi \circ \partial = \partial \circ \varphi$.
On the other hand $\varphi \circ M_b = M_b \circ \varphi, \forall b \in \mathfrak{D} \Longrightarrow \varphi \circ m_z = m_z \circ \varphi$.
Hence for $f \in K < z >$, one has $\varphi(m_z(f)) = \varphi(zf) = m_z(\varphi(f)) = z\varphi(f)$ and $\varphi(z^n f) = z^n \varphi(f)$. . As above by linearity and continuity, $\varphi(gf) = g\varphi(f), \forall g \in K < z >$. In particular $\varphi(g) = \varphi(1)g, \forall g \in K < z >$.
Setting $\varphi(1) = \sum\limits_{n \geq 0} \alpha_{0,n} z^n$, one obtains $\varphi \circ \partial(1) = \varphi(0) = 0 = \partial \circ \varphi(1) =$

$= \sum_{\geq 0}(n+1)\alpha_{0,n+1}z^n \Longrightarrow \alpha_{0,n+1} = 0, \forall n \geq 0$, and $\varphi(1) = a_{0,0} \in K$.
Hence $\varphi = a_{0,0} \cdot id$.

REMARK 3.4. The representations U^{ϖ_1} and U^{ϖ_2} are equivalent if and only if $\varpi_1 = \varpi_2$.

3.3. The completion of the Weyl algebra $A_1(K)$.

Remarks

$-(i)-$ One has three one parameter groups attached to U^{ϖ}.
Namely the group homomorphisms of $\mathfrak{D}$ into $Aut_K(K < z >)$ defined by :

-(1)- $a \to \tau_a = \exp(a\partial)$ (strong convergence)

-(2)- $b \to \exp_{\varpi}(bm_z) = M_{\exp_{\varpi}(bz)}$, the operators of multiplication by the restricted formal power series $\exp_{\varpi}(bz)$.

-(3)- $c \to \lambda_{\exp_{\varpi}(c)}$, where $\lambda_{\exp_{\varpi}(c)}$ is the linear automorphism of $K < z >$ of multiplication by the scalar $\exp_{\varpi}(c)$.

$-(ii)-$ Heisenberg commutation relation:

$$\partial \circ m_z = m_z \circ \partial + id$$

Let $\mathfrak{H} = K.\partial + K.m_z + K.id \subset \mathcal{L}(K < z >)$.

If one puts for $u, v \in \mathcal{L}(K < z >)$ the bracket $[u, v] = u \circ v - v \circ u$ one obtains, as for any associative algebra a Lie algebra structure on $\mathcal{L}(K < z >)$.

Since $[\partial, m_z] = id$, it is readily seen that $\mathfrak{H} = K.\partial + K.m_z + K.id$ is a three dimensional Lie subalgebra of $\mathcal{L}(K < z >)$ isomorphic to the K-Heisenberg Lie algebra of dimension 3.

$-(iii)-$ Matrix representation of ∂ and m_z

Let us set $\psi_n = z^n$. Then $(\psi_n)_{n\geq 0}$ is an orthonormal basis of $K < z >$.
Let $\psi_n^\star \in \mathcal{L}(K < z >, K)$ be the dual basis element such that $< \psi_n^\star, \psi_m >= \delta_{n,m}$
One has $\partial(\psi_n) = n\psi_{n-1} = n < \psi_n^\star, \psi_n > \psi_{n-1} = n\psi_n^\star \otimes \psi_{n-1}(\psi_n)$.
That is $\partial = \sum_{n\geq 0} n\psi_n^\star \otimes \psi_{n-1} = \sum_{n\geq 0}(n+1)\psi_{n+1}^\star \otimes \psi_n$.
Let us consider the canonical scalar product on $K < z >$ such that for
$f = \sum_{n\geq 0} a_n\psi_n$ and $g = \sum_{n\geq 0} b_n\psi_n$, one has $< f, g >= \sum_{n\geq 0} a_n b_n$.
Since $\partial = \sum_{\ell,j} \alpha_{\ell,j}\psi_j^\star \otimes \psi_\ell$, with $\alpha_{\ell,j} = 0$ if $(\ell, j) \notin \{n+1, n\}$ and $\alpha_{n,n+1} = n+1$, for ℓ fixed $\alpha_{\ell,j} = 0$ for $j \neq \ell + 1$; hence $\lim_{j\to+\infty} \alpha_{\ell,j} = 0$ and with respect to the scalar

product $<\ >$ the operator ∂ has an adjoint $\partial^\star = {}^t\partial \in \mathcal{L}(K < z >)$ [cf [**1**]] with ${}^t\partial = \sum_{\ell,j} \alpha_{j,\ell}\psi_j^\star \otimes \psi_\ell = \sum_{n\geq 0}(n+1)\psi_n^\star \otimes \psi_{n+1}$.
That is ${}^t\partial(\psi_n) = (n+1)\psi_{n+1}$

In the same way, one has $m_z(\psi_n) = \psi_{n+1} = \psi_n^\star(\psi_n)\psi_{n+1} = \psi_n^\star \otimes \psi_{n+1}(\psi_n)$.
Hence $m_z = \sum_{n\geq 0} \psi_n^\star \otimes \psi_{n+1}$.
One sees that m_z has an adjoint $m_z^\star = {}^t m_z = \sum_{n\geq 0} \psi_{n+1}^\star \otimes \psi_n \in \mathcal{L}(K < z >)$,
that is ${}^t m_z(\psi_n) = \psi_{n-1}, n \geq 1$ and ${}^t m_z(\psi_0) = 0$ $\square$.

3.4. The Weyl algebra $A_1(K)$.

To go straight, we shall define, following the algebraic setting in [**3**], the Weyl algebra $A_1(K)$ to be the subalgebra of $\mathcal{L}(K < z >)$ generated by $\{m_z, \partial\}$.
In fact it is the algebra of differential operators of the algebra of polynomials $K[z]$.
On the other hand, the abstract Weyl algebra is the quotient of the free algebra in two variables $K\{x, y\}$ by the two-sided ideal $\mathcal{I}$ generated by $yx - xy - 1$. And if K is of characteristic 0, then sending x on m_z and y on ∂ one obtains an isomorphism of $K\{x, y\}/\mathcal{I}$ onto $A_1(K)$.
Any $u \in A_1(K)$ can be written in the unique form $u = \sum_{i,j} \alpha_{i,j} m_z^i \partial^j$ (see for instance loc. cit.). In the sequel, we identify z^i with the operator m_z^i and then write also $u = \sum_{i,j} \alpha_{i,j} z^i \partial^j$.

Let us set $\delta^{[j]} = \dfrac{\partial^j}{j!}$. One verifies that $\delta^{[i]}\delta^{[j]} = \binom{i+j}{i}\delta^{[i+j]}$. One says that $(\delta^{[j]})_{i\geq 0}$ is an exponential sequence. For the integers $j \geq 0$ and $n \geq 0$, one has $\partial^j(z^n) = j!\binom{n}{j}z^{n-j}$ and $\delta^{[j]}(z^n) = \binom{n}{j}z^{n-j}$. Hence $\|\delta^{[j]}(z^n)\| = \left|\binom{n}{j}\right|$.
One sees that $\|\delta^{[j]}\| = \sup_{n\geq j}\left|\binom{n}{j}\right| = 1$ and $\|\partial^j\| = |j!|$.

The following statements are counterparts of results obtained some years ago by the first author and Fana Tangara ([**2**]).

Lemma 3.5. *The family $\left(m_z^i\delta^{[j]}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is an orthonormal family of $A_1(K)$ for the norm of bounded linear operators.*

Proof

-(a)-

Let $f \in K < z >$, for $i \geq 0, j \geq 0$, one has $m_z^i \circ \delta^{[j]}(f) = z^i f^{[j]}$, then
$\|m_z^i \circ \delta^{[j]}(f)\| = \|z^i \delta^{[j]}(f)\| = \|\delta^{[j]}(f)\| \leq \|f\| \Longrightarrow \|m_z^i \circ \delta^{[j]}\| = \|\delta^{[j]}\| = 1.$

-(b)-

Let $u = \sum_{i,j} \alpha_{i,j} m_z^i \partial^j = \sum_{i,j} \beta_{i,j} z^i \delta^{[j]} \in A_1(K)$ one has $\|u\| \leq \max_{i,j} |\beta_{i,j}| \|m_z^i \delta^{[j]}\| = \max_{i,j} |\beta_{i,j}|$. On the other hand $u(z^0) = u(1) = \sum_{i\geq 0} \beta_{i,0} z^i$, then $\|u(z^0)\| = \sup_{i\geq 0} |\beta_{i,0}| \leq \|u\|$ By induction, one proves that $\max_{i\geq 0} |\beta_{i,\ell}| \leq \|u\|, \forall \ell \geq 0$. One concludes that $\|u\| = \max_{i,\ell} |\beta_{i,\ell}|$. That means that $\left(m_z^i \delta^{[j]}\right)_{i,j}$ is an orthonormal family of $A_1(K)$. □

Proposition 3.6.

Let $\widetilde{A}_1(K)$ be the closure of the Weyl algebra $A_1(K)$ in the Banach algebra $\mathcal{L}(K < z >)$.

Then any element $u \in \widetilde{A}_1(K)$ can be written in the form of a unique summable family $u = \sum_{i,j} \beta_{i,j} m_z^i \delta^{[j]}$.

Moreover, one has $\|u\| = \sup_{i,j} |\beta_{i,j}|$, that is $(m_z^i \delta^{[j]})_{i,j}$ is an orthonormal basis of $\widetilde{A}_1(K)$.

Proof

This is an easy consequence of the fact that the linear basis $\left(m_z^i \delta^{[j]}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ of $A_1(K)$ is an orthonormal family in the Banach space $\mathcal{L}(K < z >)$. □

Proposition 3.7.

For $u \in \widetilde{A}_1(K)$ and $f \in K < z >$ let us set $u.f = u(f)$.
Then $K < z >$ is a left Banach module over $\widetilde{A}_1(K)$ and is topologically irreducible.

Proof

It runs as the proof of Theorem 3.2.

Proposition 3.8.

Any element $u \in \widetilde{A}_1(K)$ can be written in the unique form of convergent series

$$u = \sum_{j\geq 0} \frac{f_j}{j!} \partial^j = \sum_{j\geq 0} f_j \delta^{[j]}, \text{ with } f_j \in K < z >, \lim_{j\to+\infty} \|f_j\| = 0 \text{ and}$$
$$\|u\| = \sup_{j\geq 0} \|f_j\|$$

Proof
Applying Proposition 3.5 to any generalized differential operator $u \in \widetilde{A}_1(K)$, one has the summable sum $u = \sum_{i,j} \beta_{i,j} m_z^i \delta^{[j]}$. This means that $\lim_{i,j} |\beta_{i,j}| = 0$ along the Fréchet filter on $\mathbb{N} \times \mathbb{N}$. Or equivalently for any $j \geq 0$, $\lim_{i \longrightarrow +\infty} |\beta_{i,j}| = 0$ and $\lim_{j \longrightarrow +\infty} \sup_{i \geq 0} |\beta_{i,j}| = 0$.
Hence one has $u = \sum_{j\geq 0}\sum_{i \geq 0} \beta_{i,j} m_z^i \circ \delta^{[j]} = \sum_{j \geq 0} m_{f_j} \circ \delta^{[j]} = \sum_{j\geq 0} f_j \delta^{[j]}$, where $f_j = \sum_{i\geq 0} \beta_{i,j} z^i \in K < z >$, with $\|f_j\| = \sup_{i \geq 0} |\beta_{i,j}|$.
And one concludes that $\sup_{j\geq 0} \|f_j\| = \|u\|$. □

From the relation $\delta^{[j]} g = \sum_{s+t=j} \delta^{[s]}(g) \delta^{[t]}$ and the expansion in Proposition 3.8, one gets a formula for the expansion of the product of two elements of $\widetilde{A}_1(K)$.

With the above notations if $u = \sum_{j \geq 0} \frac{f_j}{j!} \partial^j$, one has $\|u\| = \sup_{j\geq 0} \|f_j\| = \|f_{j_0}\|$. If j_0 is the greatest integer such that $\|u\| = \|f_{j_0}\|$ then $\|f_j\| = \sup_{i\geq 0} |\beta_{i,j}| < \|f_{j_0}\|$. On the other hand let i_0 be the greatest integer $j \geq 0$ such that $\|f_{j_0}\| = |\beta_{i_0,j_0}|$, one has $|\beta_{i,j}| < \beta_{i_0,j_0}|, \forall i \geq, \forall j > j_0$ and $\|u\| = |\beta_{i_0,j_0}|$.
Considering the element $v = \sum_{i \leq i_0, j \leq j_0} \beta_{i,j} z^i \delta^{[j]}$ **of** $A_1(K)$,
one has $\|u - v\| = |\sup_{i > i_0, j > j_0} \beta_{i,j}| < \|u\|$, hence $\|u\| = \|v\|$. One says as for Tate algebra that v is a distinguished differential operator of degree (i_0, j_0).
With this in hand, one has algorithm of division by distinguished differential operators. As a consequence, following [**8**] one has

THEOREM 3.9.
The complete Weyl algebra $\widetilde{A}_1(K)$ is a simple, left noetherian ring with center K

Proof
For a proof, one can proceed as in [**8**]

THANK YOU

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